Discussion of the solutions for a system of linear equations with two variables

A. DETERMINANTS WITH TWO ROWS

$$\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases} \qquad (1)$$

The solutions to the unknowns x and y are given in the second formulas:

$$x=\frac{c_1\cdot b_2-c_2\cdot b_1}{a_1\cdot b_2-a_2\cdot b_1} \quad \text{and} \quad y=\frac{a_1\cdot c_2-a_2\cdot c_1}{a_1\cdot b_2-a_2\cdot b_1} \qquad (2)$$

For $(a_1\cdot b_2-a_2\cdot b_1)\neq 0$

for $a_1\cdot b_2-a_2\cdot b_1$ we introduce the following:

$$D=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

or

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=a_1\cdot b_2-a_2\cdot b_1$$

Analogue to that we have:

$$D_1=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}=c_1\cdot b_2-c_2\cdot b_1$$

$$D_2=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}=a_1\cdot c_2-a_2\cdot c_1$$

The number

$$D=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad (3)$$

Is called a **determinant of the system** (1),

$$D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \quad \text{i} \quad D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

(4)

respectively, the **determinant x** and **determinant y**

The expression $a \cdot d - b \cdot c$,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

Is called a **determinant of two rows.**

For the solutions to the system (1), based on (2), (3) and (4), follows:

$$x = \frac{D_1}{D}, \qquad y = \frac{D_2}{D}.$$

Example 1. $\begin{vmatrix} 3 & 4 \\ 4 & -3 \end{vmatrix} = 3 \cdot (-3) - 4 \cdot 4 = -25,$

Example 2. Solve the system of equations:

$$\begin{cases} 5x + 2y = 29 \\ 7x - 2y = 7 \end{cases}$$

Solution:

$$D = \begin{vmatrix} 5 & 2 \\ 7 & -2 \end{vmatrix} = -24, \quad D_1 = \begin{vmatrix} 29 & 2 \\ 7 & -2 \end{vmatrix} = -72, \quad D_2 = \begin{vmatrix} 5 & 29 \\ 7 & 7 \end{vmatrix} = -168,$$

$$x = \frac{D_1}{D} = \frac{-72}{-24} = 3, \qquad y = \frac{D_2}{D} = \frac{-168}{-24} = 7$$

Solution to the system is $(x, y) = (3, 7)$

We have a determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

The numbers a, b, c, d are called **elements** determinant,

$a \quad b$ i $c \quad d$ are called **rows,** while $\begin{matrix} a \\ c \end{matrix}$ i $\begin{matrix} b \\ d \end{matrix}$ … **columns** of the determinant.

The elements a and d comprise the **leading diagonal**, while c and b are a **antidiagonal** or **counterdiagonal** of the determinant

Determinants with two rows have the following properties:

1) *Determinats aren't changed if the rows become columns, and the columns become rows.*
2) *If both rows(and both columns) change their positions the determinant doesn't change its sign.*
3) *The determinant is multiplied when the elements of one of the rows or columns is multiplied*
4) *If the elements of one row(or column) are proportional with the elements of the second row(or columns), the determinant is equal to zero*
5) *If the elements of one row or column are equal to **zero**,the determinant is equal to **zero***

B. Discussion of the solutions to a system of linear equations with two variables

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

Dependent on whether the determinant of the system and the determinant of the unknowns are equal or different and not equal to zero we differentiate the following scenarios:

1^0 AIf the determiants of the system is not equal to zero, then the sytem has one and only solution

$$x = \frac{D_1}{D}, \qquad y = \frac{D_2}{D}.$$

Where D is a determinant of the system, and D_1 i D_2, are determinants of the unknowns… and for the system we say it's **defined**.

The requirement $D \neq 0$, can be represented as:

$$a_1 \cdot b_2 \neq a_2 \cdot b_1$$

or:

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

Accordingly, the system is defined, if the coefficient in front of the unknowns of one equations are proportional to the coefficients to the other equation

2^0 If he determinant of the system D is equal to **zero** and at least one of the determinants D_1 and D_2 is equal to zero, then the system has infinite number of solutions, and for it we say that its **undefined.**

From the : $D = 0$, for example, $D_1 = 0$, we have:

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \quad \text{i} \quad \frac{c_1}{c_2} = \frac{b_1}{b_2}$$

or:

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Accordingly, the system is undefined if the a_1, b_1 and c_1 from one equation are proportional to the numbers a_2, b_2 i c_2 from the second equation.

3^0 If the determinant of the system D is equal to zero, if the determinant D_1 and D_2 are not **zero** , the system has no solutions.

From the following: $D = 0$, $D_1 \neq 0$ and $D_2 \neq 0$ follows that the coefficients of one equations are proportional to the appropriate coefficients of the second equation, while the numbers c_1 and c_2 are not proportional to them.

Example. Examine the solution of the following system:

$$\begin{cases} ax + y = 1 \\ x - y = a \end{cases}$$

Determinant of the system is:

$$D = \begin{vmatrix} a & 1 \\ 1 & -1 \end{vmatrix} = -1 - a$$

1^0. If $a = -1$, then the determinant is equal to zero.
In this case the determinants of the unknowns are:

$$D_1 = \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = -1 + 1 = 0$$

$$D_2 = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 1 - 1 = 0$$

The system has infinite number of solutions.

2^0. If $a \neq 0$, then the determinant of the system is different from zero and the sytem has the solution: ($x = 1$, $y = 1 - a$).

Tasks:

1. Solve the following:

a) $\begin{cases} 2x + y = 8 \\ x - y = 4 \end{cases}$ b) $\begin{cases} 5x + 2y = 12 \\ x + 7y = 9 \end{cases}$ v) $\begin{cases} \dfrac{x+y}{3} + x = 15 \\ y - \dfrac{y-x}{5} = \dfrac{6}{5} \end{cases}$

2. Examine the solutions of the following system of equations:

a) $\begin{cases} ax + y = 2a \\ 2x + y = a \end{cases}$ b) $\begin{cases} 3ax + y = a \\ 2ax + y = b \end{cases}$